# On the Density of Triangles with Periodic Billiard Paths

fae laren
fae@faec.me

**Abstract**

We consider the classic open problem of whether every triangle has a periodic billiard path. While this has been shown for rational-angled triangles [Masur86], the irrational case remains open. Finding periodic billiard paths is an easy exercise for acute triangles, a long computer-aided case analysis when the maximum angle is at most 100° [Schwartz08], and beyond that very little is known.

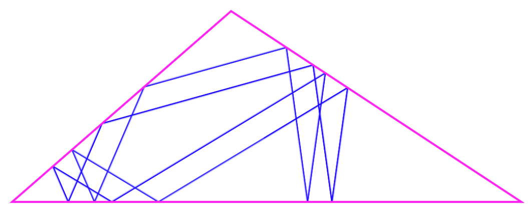
We examine the inequalities characterizing the set of triangles over which a given billiard path is periodic. We prove polynomial upper bounds on their rate of change, and use these bounds to derive positive radii within which a periodic billiard path must remain valid. We perform a computer search for periodic billiard paths on randomly selected triangles over a fixed rational grid, and use the radius bounds to show (under a Bayesian model with constant prior, assuming the uniformity of the selected grid points but otherwise unconditionally) that the likelihood that $\geq 98\%$ of all obtuse triangles admit a periodic billiard path is $> 0.99999$.

## Background

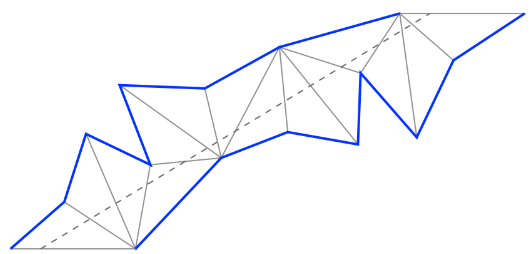
A natural way to model a billiard path on a triangle is as a straight line in the plane, along which the triangle is repeatedly reflected. A billiard path is then periodic if it ever reaches the same relative point on a triangle with the same orientation (see Figure 1). This is equivalent to the usual model, but is easier to visualize and compute with, so we will use it exclusively.

The basis on which the problem can be systematically analyzed is this: each reflection of the triangle through its edges changes the angle of the edges by integer multiples of the triangle's vertex angles. Inductively, any sequence of reflections will yield a triangle whose orientation differs by an integer linear combination of the vertex angles. If those angles are irrational, then the only way such a combination can equal zero – that is, the only way for two triangles to have the same orientation – is if the coefficients in the linear combination are also zero. That means that if a sequence of reflections produces a final triangle with the same orientation as the first, then the same will be true for *all* initial triangles, regardless of geometry. Since a periodic billiard path must pass through the interior of the reflection edges, and the vertices arising from reflection are continuous functions of the original triangle coordinates, any such path remains valid for all triangles in an open neighborhood.

This is a constructive argument: given a periodic billiard path on a particular triangle, one can derive the algebraic constraints on the triangle vertices such that a trajectory with the same combinatorial type is still a periodic path. [Schwartz08] explored this approach, exhibiting several paths and path families which remain periodic over a relatively large range, the union of which covered all triangles with maximum angle less than 100°.



**(a)** *A periodic billiard path as a ray reflected off the triangle's boundary*



**(b)** *The same path visualized by reflecting triangles along a straight line*

**Figure 1**

Unfortunately, this approach doesn't scale well: the combinatorial length of a periodic path increases with the maximum angle of the triangle. This increases the computation cost for analysis and, more importantly, dramatically decreases the radius within which the path is periodic, meaning the number of different cases required to substantially increase the current 100° bound is (probably) intractable.

### Our Results

We derive new bounds on the derivatives of the constraint functions for periodic paths. Our bounds depend polynomially on the combinatorial length of the path (i.e. the number of reflections), but not at all on the path itself. This means that given any periodic path on a triangle, and without any further analysis beyond a single evaluation of the constraint functions, we immediately have an explicit radius within which every triangle is guaranteed to have a periodic billiard path.

We apply these bounds experimentally to derive formal lower bounds on the density of triangles with periodic billiard paths, by showing that for a large fraction of triangles we can exhibit an explicit path.

# Constraint bounds

An *edge path* $E$ is a sequence $E_{1\ldots n}$ of $n$ edge indices $\{1,2,3\}$ through which a triangle $T$ is to be reflected. $\mathrm{Tri}_E(T,i)$ is the $i$th triangle in the reflected sequence when starting from $T$, defined inductively as $\mathrm{Tri}_E(T,0)=T$ and $\mathrm{Tri}_E(T,i)=$ the reflection of $\mathrm{Tri}_E(T,i-1)$ through edge $E_i$.

An edge path is *closed* if its final triangle $\mathrm{Tri}_E(T,n)$ is generically (for all $T$) of the same orientation as the first $\mathrm{Tri}_E(T,0)$. The *offset* $\overrightarrow{\mathrm{Off}}_E(T)$ of a closed edge path $E$ on triangle $T$ is the vector offset from the first triangle $\mathrm{Tri}_E(T,0)$ to the last one $\mathrm{Tri}_E(T,n)$.

Given a closed edge path, the question of whether it admits a periodic billiard path on a particular triangle can be answered by linear inequalities on the vertices of the reflected triangles: does any path parallel to $\overrightarrow{\mathrm{Off}}_E(T)$ remain strictly inside the union of the reflected triangles $\cup_i \mathrm{Tri}_E(T,i)$ (as in Figure 1b)? We choose to think of edge paths as starting from the base edge of an initial oriented triangle and proceeding upward, so we call the ordered vertices bounding the reflections the *left boundary* and *right boundary*, and denote the sequences by $(\mathrm{Left}_i)$ and $(\mathrm{Right}_i)$. While their precise coordinates depend on $T$, combinatorially they are a function only of the edge path itself. In particular, the length of each sequence depends only on $E$.

The *constraint functions* $\psi_{i,j}$ for a closed edge path $E$ on a triangle $T$ are polynomials computing (a positive multiple of) the margin between the left and right boundary vertices when moving along the path's offset:

$$\psi_{i,j}(E,T) = \langle \overrightarrow{\mathrm{Off}}_E(T)^{\perp}, \mathrm{Left}_i - \mathrm{Right}_j \rangle \qquad (1)$$

where $i$ and $j$ range over the lengths of the left and right boundary sets, and $\overrightarrow{\mathrm{Off}}_E(T)^{\perp}$ is the rotation of $\overrightarrow{\mathrm{Off}}_E(T)$ by $\pi/2$. There is a periodic billiard path on $T$ through $E$ if and only if $\psi_{i,j}(E,T)>0$ for all $i$ and $j$.

Given $E$ and a $T$ with rational vertex coordinates, the constraint functions can be evaluated explicitly to determine if $E$ yields a periodic billiard path on $T$. Given only $E$, variables can be substituted for $T$'s coordinates to yield generic polynomial constraints that are satisfied only on the set of triangle coordinates for which $E$ gives a periodic billiard path. This is, in outline, the method underlying the results of [Schwartz08].

To simplify formulas in what follows we without loss of generality consider triangles whose longest edge (the base) lies on a fixed unit interval, so that any two triangles differ only in the coordinates of their apex. We then have:

**Theorem 1.** *Given triangles $T_1, T_2$ with apexes $a_1, a_2$ and a closed edge path $E$ of length $n$, let $\ell_{min}$ be the length of the shortest edge of $T_1$ and define $\psi_{i,j}$ as in Equation 1. Then*

$$\left|\psi_{i,j}(E,T_1) - \psi_{i,j}(E,T_2)\right| \le \frac{(n+2)^3\|a_1-a_2\|}{8\ell_{min}} \qquad (2)$$

*for all $i$ and $j$.*

**Corollary 1.** *Given triangle $T_1$ with apex $a_1$ having a periodic billiard path along a closed edge path $E$ of length $n$, let $\ell_{min}$ be the length of the shortest edge of $T_1$, and set $\psi_{min} = min_{i,j}[\psi_{i,j}(E,T)]$. Then every triangle $T_2$ with apex $a_2$ such that*

$$\|a_1-a_2\| \le \frac{8\ell_{min}\psi_{min}}{(n+2)^3} \qquad (3)$$

*also has a periodic billiard path along $E$.*

Given a periodic billard path on a triangle with rational vertices, Corollary 1 gives an explicit radius within which the apex of the triangle can be perturbed while preserving the billiard path.

# Experimental Results

We now wish to obtain a lower bound on the fraction of obtuse triangles that admit periodic billiard paths. As above, we set the longest edge to be a fixed unit-length base. By symmetry, we also assume the shortest edge is the one lying clockwise of the base. Our problem space is thus the set of possible apexes within a radius-1/2 quarter-circle. With this parameterization, the "fraction" of obtuse triangles with periodic paths is taken to mean the relative measure within that quarter-circle of the set of apexes admitting a periodic path.

Our approach is to fix a rational grid covering the problem space and select apexes uniformly at random from the grid, then search for periodic billiard paths for the chosen apexes. If a path is found, and Corollary 1 shows that it remains valid up to a radius larger than the grid spacing, then all the triangles nearest to that grid point are guaranteed to have periodic billiard paths, and we call that apex a success. A high rate of success gives us a Bayesian lower bound on the probability that in fact a high proportion of all obtuse triangles admit periodic billiard paths.

| | |
|---|---|
| Grid spacing | $2 \times 10^{-14}$ |
| Apex count | 2500 |
| Success count | 2479 |

| Path length | |
|---|---|
| Min | 14 |
| Max | 28572 |
| Avg | 288.1 |

| Path validity radius | |
|---|---|
| Min | $1.45 \times 10^{-14}$ |
| Max | $1.3 \times 10^{-3}$ |
| Mult. Avg | $8.6 \times 10^{-7}$ |

**Table 1:** *Experimental results*

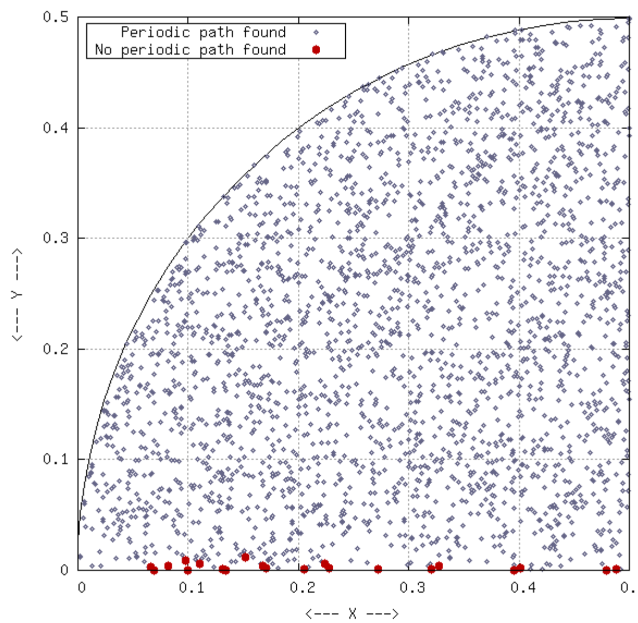



**Figure 2:** *A scatter plot of the trial outcomes. Successful points denote a cyclic billiard path that is valid up to a positive radius greater than the spacing of the apex grid.*

Once a periodic path is found, it is easy to independently verify its correctness and radius, so the probabilities we derive are conditioned only on the source of randomness used to generate the apex set. We used `libbsd`'s `arc4random` running on Ubuntu Linux 17.04, which we consider robust enough for problems of this nature, but it would be straightforward to reproduce the experiment using any desired source.

Our numeric results are in Table 1, with a scatter plot in Figure 2. Given the preceding, we consider this to be strong empirical evidence for the assertion that most obtuse triangles admit periodic billiard paths. Specifically, if we assume the uniformity of our test coordinates and compute the conditional probability of the success rate we have:

**Confidence Bound 1.** *Assuming a constant prior on the fraction of obtuse triangles that admit a periodic billiard path, and given an experimental outcome at least as good as the one in Table 1, the likelihood that* $\geq 98\%$ *of obtuse triangles admit a periodic billiard path is* $> 0.99999$.

Both the likelihood and the proportion of triangles may be readily increased by repeating the experiment with a higher trial count and search depth.

# Proof Sketch

We now outline the proof of Theorem 1. Set $T = T_1$, $a = a_1$. As above, let $(\text{Left}_i)$ and $(\text{Right}_i)$ denote the left and right boundaries of $T$'s reflections along $E$.

If there is a periodic billiard path, it must lie parallel to the edge path's offset $\overrightarrow{\text{Off}}_E(T) = \text{Left}_{\text{final}} - \text{Left}_0$. This means it lies perpendicular to $\overrightarrow{\text{Off}}_E(T)^\perp$, so given points $p_1$ and $p_2$ we can measure their mutual distance orthogonal to the offset via the dot product $\langle \overrightarrow{\text{Off}}_E(T)^\perp, p_2 - p_1 \rangle$, which is positive if and only if $p_2$ lies to the left of $p_1$ when traveling in the direction of $\overrightarrow{\text{Off}}_E(t)$. From this we get the constraint functions $\psi_{i,j}$ of Equation 1.

Let us telescope the boundary representation of the offset:

$$\begin{aligned} \overrightarrow{\text{Off}}_E(T) &= \text{Left}_{\text{final}} - \text{Left}_0 \\ &= \sum_{i=1}^{\text{final}} (\text{Left}_i - \text{Left}_{i-1}) \end{aligned} \tag{4}$$

The elements of this sum are edge vectors along the boundary. Focusing on these edges, we define:

$$\begin{aligned} \overrightarrow{\text{Left}}_i &= \text{Left}_i - \text{Left}_{i-1} \\ \overrightarrow{\text{Right}}_i &= \text{Right}_i - \text{Right}_{i-1} \end{aligned} \tag{5}$$

Every boundary vertex can then be expressed as a sum of edge vectors from the first points:

$$\begin{aligned} \text{Left}_i &= \text{Left}_0 + \sum_{j \leq i} \overrightarrow{\text{Left}}_j \\ \text{Right}_i &= \text{Right}_0 + \sum_{j \leq i} \overrightarrow{\text{Right}}_j \end{aligned} \tag{6}$$

Since the operation $\perp$ is linear, it commutes with this form, so we also have:

$$\overrightarrow{\text{Off}}_E(T)^\perp = \sum_i \overrightarrow{\text{Left}}_i^\perp = \sum_i \overrightarrow{\text{Right}}_i^\perp \tag{7}$$

Combining these gives us the following expression for the constraint functions:

$$\begin{aligned}\psi_{i,j}(E,T) = &\left\langle \sum_k \overrightarrow{\mathrm{Left}}_k^\perp, \sum_{k\le i} \overrightarrow{\mathrm{Left}}_k \right\rangle \\ &+ \left\langle \sum_k \overrightarrow{\mathrm{Right}}_k^\perp, \sum_{k\le j} \overrightarrow{\mathrm{Right}}_k \right\rangle\end{aligned} \tag{8}$$

This suggests many simplifications using the fact that the inner product of a vector with its orthogonal is zero. All we will use for now is that (since both $i$ and $j$ are bounded by the number of reflections $n$) we can expand both sides of the inner products by distributivity into a sum

$$\psi_{i,j}(E,T) = \sum_k^N \langle v_k^\perp, w_k \rangle \tag{9}$$

where the $v_k, w_k$ are elements of $\overrightarrow{\mathrm{Left}}$ or $\overrightarrow{\mathrm{Right}}$ and $N$ is at most $(n+2)^2/4$. This is not completely trivial: an $n^2/2$ bound is immediate, but the extra $1/2$ factor is obtained by more careful analysis of path length, which we omit here.

For the rest, we can rewrite Equation 9 in angle form:

$$\psi_{i,j}(E,T) = \sum_k^N \|v_k\| \|w_k\| \cos(\theta_k) \tag{10}$$

where $\theta_k$ is the angle between $v_k^\perp$ and $w_k$. Recall that the angle of edges produced by repeated reflection varies by integer combinations of the triangle's vertex angles $\alpha_{\{1,2,3\}}$, so $\theta_k = \pi/2 + \sum_i c_{k,i}\alpha_i$ for integers $(c_{k,i})$. Induction shows the difference in any two edge angles is at most $n/2$ individual rotations, so we may choose $\sum_i |c_{k,i}| \le n/2$.

Now vary the apex $a$ by an infinitesimal offset $\overrightarrow{d(a)}$, and let $d(r) = \left\|\overrightarrow{d(a)}\right\|$ be the (positive) infinitesimal change in distance from $a$. Let $d(\alpha_i)$ be the change in angle $i$ of $T$, and let $d(\alpha)_{\max}$ be the largest-magnitude change in any angle of $T$. Then the change of angle $\theta_k$ can also be bounded:

$$|d(\theta_k)| \le \frac{n|d(\alpha)_{\max}|}{2} \tag{11}$$

by the coefficient bound above.

Either of the base angles have their variation $|d(\alpha_i)|$ bounded by $\frac{d(r)}{\ell_{\min}}$, so the change in the apex angle $|d(\alpha_3)|$ is at most $\frac{2d(r)}{\ell_{\min}}$. Substituting this for $d(\alpha)_{\max}$ gives:

$$|d(\theta_k)| \le \frac{nd(r)}{\ell_{\min}} \tag{12}$$

We now differentiate Equation 10 and substitute these inequalities, obtaining:

$$\begin{aligned} d(\psi_{i,j}) &\le d(r)\left(\frac{n+2}{\ell_{\min}}\right)\left(\frac{(n+2)^2}{4}\right) \\ &= \frac{d(r)(n+2)^3}{4\ell_{\min}} \end{aligned} \tag{13}$$

Again there is a missing factor of 1/2 in this argument. The 8 in the denominator in the full theorem comes from rewriting the preceding expressions to omit the apex angle $\alpha_3$. This can be done without breaking any of the bounds, but it requires lengthier bookkeeping, so we again omit it here.

## Conclusion

Our results give the first quantitative empirical evidence that a high proportion of obtuse triangles have periodic billiard paths. Though our methods can't extend to the full conjecture that *all* triangles do, some improvements and extensions still suggest themselves:

- Our experiments have yielded a larger and more diverse set of example billiard paths than was previously available. Examination of these paths reveal common structures that in some cases are amenable to analysis; exploring these might allow better heuristics for finding periodic paths, or (if one is optimistic) even unconditional algorithms for some classes of triangles.
- Observations during testing lead us to believe that the remaining failure cases in our dataset are limited by the grid density we chose: most of them have periodic billiard paths whose radius is too low. A denser grid, or one that grew progressively denser as it approached the base, would probably have a greater success rate for very narrow triangles.
- Our heuristic for finding periodic billiard paths was relatively naive, essentially just testing random vectors from a promising distribution. While this was enough to succeed on most inputs, it seems likely that more principled selection would do even better.

## References


[Masur86] Masur, Howard. Closed Trajectories for Quadratic Differentials with an Application to Billiards. *Duke Math. J.* 53 (1986), 307–313.

[Schwartz08] Schwartz, Richard Evan. Obtuse Triangular Billiards II: 100 Degrees Worth of Periodic Trajectories. *Experiment. Math.* 18 (2009), no. 2, 137–171.